JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 116, No. 3, pp. 559–590, March 2003 (© 2003)

An Interior-Point Method for a Class of Saddle-Point Problems[1]

B. V. Halldórsson[2] and R. H. Tütüncü[3]

Communicated by F. A. Potra

Abstract. We present a polynomial-time interior-point algorithm for a class of nonlinear saddle-point problems that involve semidefiniteness constraints on matrix variables. These problems originate from robust optimization formulations of convex quadratic programming problems with uncertain input parameters. As an application of our approach, we discuss a robust formulation of the Markowitz portfolio selection model.

Key Words. Interior-point methods, robust optimization, portfolio optimization, saddle-point problems, quadratic programming, semidefinite programming.

1. Introduction

We study the solution of nonlinear saddle-point problems that arise from a robust optimization formulation of convex quadratic programming problems whose input parameters are uncertain. We develop interior-point methods to solve these problems in polynomial time.

Consider a convex quadratic programming (QP) problem given in the following form:

$$\min_{x} \quad c^T x + (1/2)x^T Qx, \tag{1a}$$

$$\text{s.t.} \quad Ax \geq b, \tag{1b}$$

[1]This research was supported in part by NSF Grants DMS-9706950 and CCR-9875559 and by the Center for Nonlinear Analysis at Carnegie Mellon University.

[2]Computer Scientist, Informatics Research, Celera Genomics, Rockville, Maryland. Most of this work was done while the author was a Graduate Student at the Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania.

[3]Associate Professor, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania.

0022-3239/03/0300-0559/0 © 2003 Plenum Publishing Corporation

where

$$A \in \mathbb{R}^{m \times n}, \qquad b \in \mathbb{R}^m, \qquad c \in \mathbb{R}^n, \qquad Q \in \mathcal{S}^n_+$$

are the input data and $x \in \mathbb{R}^n$ are the decision variables. $\mathcal{S}^n$ denotes the space of $n \times n$ symmetric matrices and $\mathcal{S}^n_+$ denotes the space of symmetric positive-semidefinite n-dimensional matrices. We will denote the feasible set by χ, i.e.,

$$\chi = \{x: Ax \geq b\},$$

and assume that it is nonempty to avoid trivial cases.

In many practical instances of the QP problem (1), some or all components of the data are uncertain. They are either unknown at the time of the problem formulation/solution and will be realized only in the future, or there are some intrinsic restrictions that makes it impossible to compute or measure these quantities exactly. Arguably, the most common approach toward handling such uncertainties is to just ignore them, substitute some highly likely representative quantities for the uncertain components of the data, and then treat the problem as a deterministic problem with no uncertainty. Once a solution is obtained for this representative problem, one may study the stability properties of this solution using the techniques of sensitivity analysis.

Stochastic programming offers a more sophisticated approach. This approach requires a probability distribution on the uncertain parameters and replaces the uncertain quantities with their expected values in the optimization model. As such, stochastic programming lays a great responsibility on the modeler who has to determine a probability distribution for the model and may generate misleading results when the distribution is an inaccurate representation of the true nature of the model.

Here, we will focus on the robust optimization approach, which was studied recently by Ben-Tal and Nemirovski in Refs. 1 and 2 among others. This approach is more conservative and is applicable particularly in cases where the modeler cannot determine accurately or predict the input data but seeks a solution that will be desirable in all possible realizations of this data, i.e., when a best worst-case performance is sought.

We study the case where the uncertainty is in the objective function of the QP and assume that the constraints are known and certain. In Section 4, we describe an application problem that motivates these assumptions. We describe the uncertainty set for the Q matrix as the intersection of a hyperrectangle of the form $\{Q: L \leq Q \leq U\}$, where the inequalities are componentwise and the cone of positive-semidefinite matrices. This particular choice of the uncertainty set is motivated by the problems where the Q

matrix is known to be positive semidefinite, e.g., when Q is a covariance matrix, but its components have to be estimated.

The use of intervals appears very natural in modeling uncertainty. The modeler may choose to look at several different scenarios, corresponding to the best and worst cases. Different methods may be used to estimate the parameters, possibly by different modelers. Different data sets can be used, and the L and U matrices correspond to the maximum and minimum of those estimations. Alternatively, the estimation can be done only using confidence intervals. Furthermore, our approach can be applied to cases where the uncertainty set is not given by intervals. Indeed, our techniques and results generalize readily to other types of uncertainty sets, as long as these sets admit a self-concordant barrier function; see Ref. 3 for a definition.

In a related work, Ben-Tal and Nemirovski apply the robust optimization approach to quadratically constrained convex quadratic programming problems; see Ref. 2. They focus on uncertainty in the constraints which can handle the objective function uncertainty after a straightforward transformation. However, they model the Q matrix as given by the equation $Q = AA^T$, and then place the uncertainty description on the A matrix. When one has an uncertainty description for only the Q matrix, transforming this set into an uncertainty description on the Cholesky-like factors of the Q matrix can be a very difficult task, if not impossible. Also, after we finished an earlier draft of the current manuscript, we became aware of ongoing work by Lobo and Boyd; see Ref. 4. This paper studies a problem similar to ours, but is focused more on the modeling issues and does not contain a complexity estimate of the corresponding algorithm or a proof of polynomiality.

Our work is also related to Nemirovski's work in Ref. 5 on self-concordant convex-concave functions. In fact, our algorithm and analysis can be regarded as a simplification of the Nemirovski algorithm and analysis for a less general class of saddle-point problems. While the barrier method in Ref. 5 requires an inner algorithm between the updates of the barrier parameter, we just take a single step of the Newton method. Like Nemirovski, we achieve a polynomial bound on the number of iterations of our short-step path-following algorithm.

The remainder of this paper is organized as follows. In Section 2, we present a robust optimization problem and discuss different formulations of it, including a formulation in the form of a saddle-point problem. We introduce also measures of proximity of the variables to the set of saddle points and the so-called central path. In Section 3, we describe a short-step interior-point algorithm and prove a polynomial complexity result for the algorithm. In Section 4, we provide an application stemming from a robust formulation of the well-known Markowitz model for portfolio optimization. Finally, we conclude in Section 5.

2. Problem Formulation

In the robust optimization approach to the solution of uncertain optimization problems, one takes the conservative viewpoint that the realization of the data will be determined by an adversary who will choose the worst-data instance for a given set of decision variables. In this setting, the task of the modeler is to choose the values for the decision variables that have the best worst-case performance.

Recall the quadratic programming problem given in (1). We will consider an environment where the objective function is uncertain but is known to be a convex function of the variables. For example, if the matrix Q in (1) represents a covariance matrix, we would know that it is positive semidefinite, and therefore that the objective function is convex, even if its entries are not known. We will describe an important example of this scenario in Section 4. We model the uncertainty in the objective function using the following uncertainty set:

$$\begin{aligned}\mathscr{U} &:= \mathscr{U}(c^L, c^U, Q^L, Q^U)\\ &:= \{(c, Q) \in \mathbb{R}^n \times \mathscr{S}^n : c^L \leq c \leq c^U,\ Q^L \leq Q \leq Q^U,\ Q \succcurlyeq 0\}, \qquad (2)\end{aligned}$$

where c^L, c^U are given n-dimensional real vectors satisfying $c^L < c^U$, and where Q^L, Q^U are in $\mathscr{S}^n$ and satisfy $Q^L < Q^U$. As mentioned above, $Q \succcurlyeq 0$ means $Q \in \mathscr{S}^n_+$. We will assume that $\mathscr{U}$ has a nonempty interior. When the objective function of a quadratic program is unknown or uncertain, the input parameters need to be estimated. Our particular choice for the uncertainty set with lower and upper bounds on the input parameters provides the modeler with a simple and natural model of uncertainty.

Note that, since Q^L and Q^U are symmetric and since Q is restricted to be symmetric, the matrix inequalities

$$Q^L \leq Q \leq Q^U$$

can be represented with $n(n+1)$ componentwise scalar inequalities say, for the upper triangular portions of these symmetric matrices. In other words, $Q^L \leq Q$ is a short-hand notation for

$$Q^L_{ij} \leq Q_{ij}, \qquad 1 \leq i \leq j \leq n,$$

and similarly for $Q \leq Q^U$. We will refer occasionally to the following projections of the set $\mathcal{Y}$:

$$\mathcal{Y}^c := \{c \in \mathbb{R}^n : c^L \leq c \leq c^U\},$$

$$\mathcal{Y}^Q := \{Q \in \mathcal{S}^n : Q^L \leq Q \leq Q^U,\ Q \succeq 0\}.$$

In what follows, we will denote the pair (c, Q) with y. Given $x \in \mathcal{X}$ and $y = (c, Q) \in \mathcal{Y}$, let

$$\begin{aligned}\phi(x, y) &= \phi(x, c, Q)\\ &:= c^T x + (1/2)\, x^T Q x\\ &= c^T x + (1/2)\, xx^T \cdot Q \qquad (3)\end{aligned}$$

denote the value of the objective function. For symmetric matrices A and B, $A \cdot B$ denotes the standard inner product, i.e.,

$$A \cdot B = \operatorname{trace}(AB) = \sum_{i,j} A_{ij} B_{ij}.$$

Note that the ϕ is a quadratic function of x and is a linear function of c and Q. For fixed $\hat{x} \in \mathcal{X}$ and fixed $\hat{y} = (\hat{c}, \hat{Q}) \in \mathcal{Y}$, we will use the following notation:

$$\phi_{\hat{x}}(y) := \phi(\hat{x}, y), \qquad \phi_{\hat{y}}(x) := \phi(x, \hat{y}).$$

For a given vector of decision variables x, we will denote the worst-case realization of the objective function by

$$f(x) := \max_{(c, Q) \in \mathcal{Y}} \phi(x, c, Q) \qquad (4)$$

$$= \max_{c \in \mathcal{Y}^c} c^T x + \max_{Q \in \mathcal{Y}^Q} (1/2)\, xx^T \cdot Q \qquad (5)$$

$$=: f^c(x) + f^Q(x). \qquad (6)$$

Note that both $\mathcal{Y}^c$ and $\mathcal{Y}^Q$ are compact sets and that the objective function for each term in (5) is linear. Therefore, the optimal values are achieved and the use of max rather than sup is justified. The first term in (5) is a linear programming problem. It is easy to see that

$$f^c(x) = \sum_i x_i^+ c_i^U - \sum_i x_i^- c_i^L.$$

Here,

$$x^+ = \max\{0, x\} \quad \text{and} \quad x^- = \max\{0, -x\}.$$

The second term in (5) is a semidefinite programming problem, which has been extensively studied in recent years; see e.g. Refs. 6–8.

For future reference, we construct now the duals of the two optimization problems in (5). For a fixed x, let $\delta^L \in \mathbb{R}^n$ and $\delta^U \in \mathbb{R}^n$ be the dual variables corresponding to the constraints $c \geq c^L$ and $c \leq c^U$. Then, the dual of the linear program in (5) is the following:

$$\min_{\delta^L, \delta^U} \quad -(c^L)^T \delta^L + (c^U)^T \delta^U, \tag{7a}$$

$$\text{s.t.} \quad -\delta^L + \delta^U = x, \tag{7b}$$

$$(\delta^L, \delta^U) \geq 0. \tag{7c}$$

Similarly, letting V^L and V^U be the dual variables corresponding to the constraints $Q \geq Q^L$ and $Q \leq Q^U$ in the semidefinite program in (5), and using the explicit dual slack variable Z, we can represent the dual of the semidefinite program that defines $f^Q(x)$ as follows:

$$\min_{V^L, V^U, Z \in \mathcal{S}^n} \quad -Q^L \cdot V^L + Q^U \cdot V^U, \tag{8a}$$

$$\text{s.t.} \quad -V^L + V^U - Z = (1/2)\, xx^T, \tag{8b}$$

$$(V^L, V^U) \geq 0, \qquad Z \succcurlyeq 0. \tag{8c}$$

We would like to find a robust solution to (1), i.e., a solution that minimizes $f(x)$,

$$\min_{x \in X} f(x) = \min_{x \in \mathcal{X}} \left\{ \max_{(c, Q) \in \mathcal{U}} \phi(x, c, Q) \right\}. \tag{9}$$

We denote the best allocation of decision variables for a given realization of the objective function by

$$g(c, Q) := \min_{x \in \mathcal{X}} \phi(x, c, Q). \tag{10}$$

For fixed c and Q, the minimization problem on the right-hand-side of (10) is a convex quadratic programming (QP) problem, provided that Q is a positive semidefinite matrix. For simplicity, we will assume that $\mathcal{X}$ is bounded. This assumption is justified in most real life applications including the example that we consider in Section 4. It can be enforced by adding box constraints of the form $-M \leq x_i \leq M$ with a large positive M, if necessary. With the boundedness assumption, the minimum is attained in the definition (10). Using the dual variables λ corresponding to the constraints $Ax \geq b$, we can construct the following dual of this QP:

$$\max_{x, \lambda} \quad b^T \lambda - (1/2)\, x^T Q x, \tag{11a}$$

$$\text{s.t.} \quad A^T \lambda - Qx = c, \tag{11b}$$

$$\lambda \geq 0. \tag{11c}$$

Then, the adversary that tries to choose the worst possible realization of the objective function would need to solve the following problem:

$$\max_{(c,\,Q)\in\mathscr{Y}} g(c, Q) = \max_{(c,\,Q)\in\mathscr{Y}} \left\{\min_{x\in\mathscr{X}} \phi(x, c, Q)\right\}. \tag{12}$$

Problems (9) and (12) are duals of each other; we will refer to (9) as the primal and to (12) as the dual. The proof of the following weak duality result is straightforward.

Lemma 2.1. For any $x\in\mathscr{X}$ and $(c, Q)\in\mathscr{Y}$, the following inequality holds:

$$g(c, Q)\leq f(x). \tag{13}$$

Consider the epigraphs of the functions f and $-g$. Using the standard argument that a function is convex if and only if its epigraph is a convex set, the following observation is easily proved.

Lemma 2.2. $f(x)$ defined in (4) is a convex function of x; $g(c, Q)$ defined in (10) is a concave function of (c, Q).

Using Lemma 2.2, we can characterize the primal-dual pair of problems (9) and (12) as convex optimization problems. However, the objective functions f and g need not be smooth functions of their arguments. One possibility to approach these problems is to use the techniques of nonsmooth optimization such as the gradient bundle algorithms in Ref. 9 or the analytic center methods in Ref. 10. Alternatively, using the duals (7) and (8) to replace $f^c(x)$ and $f^Q(x)$ in (6), we can also represent (9) as a nonlinear semidefinite programming problem. However, we prefer a saddle-point approach as outlined below.

The function $\phi(x, y)$ is convex–concave; i.e., for any fixed $\hat{y}\in\mathscr{Y}$, the restricted function $\phi_{\hat{y}}(x)$ is convex in x and, for any fixed $\hat{x}\in\mathscr{X}$, the restricted function $\phi_{\hat{x}}(y)$ is concave (in fact, linear) in y. Since $\mathscr{X}$ and $\mathscr{Y}$ are assumed to be nonempty and bounded, the following lemma follows immediately from Theorem 37.3 on pp. 392–393 of Ref. 11.

Lemma 2.3. The optimal values of (9) and (12) are equal.

Theorem 37.3 of Ref. 11 actually tells us more; it says that the function ϕ has a saddle point. This observation allows us to formulate each of problems (9) and (12) as a saddle-point problem. In this representation, the function $\phi(x, y) = \phi(x, c, Q)$ acts as the saddle function and we look for $\bar{x}\in\mathscr{X}$ and $(\bar{c}, \bar{Q})\in\mathscr{Y}$ such that

$$\text{(SPP)}\quad \phi(\bar{x}, c, Q)\leq \phi(\bar{x}, \bar{c}, \bar{Q}), \leq \phi(x, \bar{c}, \bar{Q}), \quad \forall x\in\mathscr{X}, (c, Q)\in\mathscr{Y}. \tag{14}$$

This representation is important in two respects. First, it shows that we can approach the robust optimization problem using the rich literature on saddle-point problems. For example, Sun, Zhu, and Zhao studied moderately nonlinear saddle-point problems and developed polynomial time algorithms in Ref. 12, while Nemirovski developed a self-concordance theory for saddle-point problems in Ref. 5. Second, the saddle-point formulation gives rise to a measure which controls the progress of an algorithm that we will introduce later in the paper.

2.1. Optimality Conditions. In this section, we discuss the optimality conditions that characterize the saddle points of the function ϕ. Let $(\bar{x}, \bar{c}, \bar{Q})$ be a saddle-point of the function ϕ with $\bar{x} \in \mathscr{X}$ and $\bar{y} = (\bar{c}, \bar{Q}) \in \mathscr{Y}$. As mentioned above, with our assumptions that $\mathscr{X}$ and $\mathscr{Y}$ are nonempty and bounded, such a point is guaranteed to exist. From the definition of a saddle point, we have that (i) $\bar{x}$ minimizes $\phi_{\bar{y}}(x)$ over $\mathscr{X}$ and that (ii) $\bar{y}$ maximizes $\phi_{\bar{x}}(y)$ over $\mathscr{Y}$. The Karush–Kuhn–Tucker conditions for these two problems, which are both necessary and sufficient because of the convexity properties of respective functions, lead to a set of conditions that characterize saddle points of ϕ.

From (i), we have that there exist Lagrange multipliers $\lambda \in \mathbb{R}^m$ such that

$$\bar{Q}\bar{x} - A^T\lambda + \bar{c} = 0, \tag{15a}$$

$$A\bar{x} - b \geq 0, \qquad \lambda \geq 0, \tag{15b}$$

$$(A\bar{x} - b) \circ \lambda = 0. \tag{15c}$$

Above, $A \circ B$ denotes the Hadamard product; i.e., for matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimensions,

$$A \circ B = C = [c_{ij}], \qquad \text{where } c_{ij} = a_{ij} b_{ij}.$$

From (ii), we conclude that there exist Lagrange multipliers $\delta^L \in \mathbb{R}^n$ and $\delta^U \in \mathbb{R}^n$ such that

$$-\delta^L + \delta^U - \bar{x} = 0, \tag{16a}$$

$$\bar{c} - c^L \geq 0, \qquad c^U - \bar{c} \geq 0, \tag{16b}$$

$$(\delta^L, \delta^U) \geq 0, \tag{16c}$$

$$(\bar{c} - c^L) \circ \delta^L = 0, \tag{16d}$$

$$(c^U - \bar{c}) \circ \delta^U = 0, \tag{16e}$$

and there exist multipliers $V^L, V^U, Z \in S^n$ such that

$$-V^L + V^U - Z - (1/2)\, \bar{x}\, \bar{x}^T = 0, \tag{17a}$$

$$\bar{Q} - Q^L \geq 0, \qquad Q^U - \bar{Q} \geq 0, \tag{17b}$$

$$(V^L, V^U) \geq 0, \qquad \bar{Q} \succcurlyeq 0, \qquad Z \succcurlyeq 0, \tag{17c}$$

$$(\bar{Q} - Q^L) \circ V^L = 0, \qquad (Q^U - \bar{Q}) \circ V^U = 0, \tag{17d}$$

$$\bar{Q}Z = 0. \tag{17e}$$

A remark about our notation is in order. As we mentioned above, because of symmetry considerations, each of the systems $Q \geq Q^L$ and $Q \leq Q^U$ can be represented using $n(n+1)/2$ componentwise constraints. Therefore, the corresponding Lagrange multiplier vectors should both lie in $\mathbb{R}^{n(n+1)/2}$. However, for ease of notation, we will represent them as n-dimensional symmetric matrices and let $V^L_{ij} = V^L_{ji}$, etc. Note that the space $\mathscr{S}^n$ has the same dimension as $\mathbb{R}^{n(n+1)/2}$. As a result, instead of writing the nonnegativity and complementarity constraints in componentwise form as

$$\begin{aligned}
&V^L_{ij} \geq 0, && 1 \leq i \leq j \leq n,\\
&V^U_{ij} \geq 0, && 1 \leq i \leq j \leq n,\\
&(\bar{Q}_{ij} - Q^L_{ij}) \circ V^L_{ij} = 0, && 1 \leq i \leq j \leq n,\\
&(Q^U_{ij} - \bar{Q}_{ij}) \circ V^U_{ij} = 0, && 1 \leq i \leq j \leq n,
\end{aligned}$$

we can use the short-hand matrix form as in (17). We recall that, for symmetric matrices A and B, $A \geq 0$ represents membership in $\mathbb{R}^{n(n+1)/2}_+$, the cone of n-dimensional symmetric matrices with nonnegative entries, unlike the inequality $B \succcurlyeq 0$ which denotes membership in $\mathscr{S}^n_+$, the cone of n-dimensional positive-semidefinite symmetric matrices; also, $A \circ B = 0$ represents $n(n+1)/2$ equations, not n^2 equations.

The optimality conditions (15)–(17) are similar to the optimality conditions that one encounters in standard linear and semidefinite programming problems; see e.g. Ref. 6. However, standard methods for such problems cannot be applied here without modification because of the nonlinearity in the x variables in (17a) and the cross-product term Qx in (15a). Fortunately, this is a mild form of nonlinearity, and we will discuss later how self-concordant barrier functions regularize the effects of this nonlinearity on the variables.

2.2. Central Path. To find a saddle point of the function ϕ, we will follow a barrier approach. That is, we will consider a barrier function that combines the self-concordant barriers on the convex sets $\mathscr{X}$ and $\mathscr{Y}$ with the saddle function ϕ; we will determine near saddle points for the combined barrier function; then, by gradually increasing the weight of the function ϕ, we will approach a saddle point for this function.

Let $\mathscr{X}^0$ and $\mathscr{Y}^0$ denote the interiors of the sets $\mathscr{X}$ and $\mathscr{Y}$, which we assume to be nonempty. Consider the following barrier functions for the sets $\mathscr{X}$ and $\mathscr{Y}$:

$$F(x) = -\sum_{i=1}^{m} \log[Ax-b]_i, \qquad \forall x \in \mathscr{X}^0, \tag{18}$$

$$\begin{aligned} G(y) = &-\sum_{j=1}^{n} \log(c_j^U - c_j) - \sum_{j=1}^{n} \log(c_j - c_j^L) - \sum_{1\le i\le j\le n} \log(Q_{ij}^U - Q_{ij}) \\ &- \sum_{1\le i\le j\le n} \log(Q_{ij} - Q_{ij}^L) - \log\det(Q), \qquad \forall y = (c, Q) \in \mathscr{Y}^0. \end{aligned} \tag{19}$$

The notion of self concordance was introduced by Nesterov and Nemirovski, and their book (Ref. 3) is the definitive reference on the subject. It is verified easily that $F(x)$ is a self-concordant barrier for $\mathscr{X}$ with parameter m and that $G(y)$ is a self-concordant barrier for $\mathscr{Y}$ with parameter n^2+4n. In fact, one can show that $G(y)$ is an a-self-concordant barrier for $\mathscr{Y}$ for all $a \ge n(n+5)/2$, but we will use the more obvious parameter n^2+4n in our discussion.

For $t \ge 0$, consider the following saddle-barrier function:

$$\phi_t(x, y) = \phi_t(x, c, Q) := t\phi(x, y) + F(x) - G(y). \tag{20}$$

We look for saddle points of the function ϕ_t, i.e., for $x_t \in \mathscr{X}^0$ and $y_t = (c_t, Q_t) \in \mathscr{Y}^0$ such that

$$(\mathrm{SPP}_t) \quad \phi_t(x_t, y) \le \phi_t(x_t, y_t) \le \phi_t(x, y_t), \qquad \forall x \in \mathscr{X}^0, y \in \mathscr{Y}^0. \tag{21}$$

Note that $F(x)$ and $G(y)$ are convex functions of their arguments. Therefore, $\phi_t(x, y)$ is a convex–concave function. Using an argument identical to the one that precedes the statement of Lemma 2.3 and using Theorem 37.3 in Ref. 11, we conclude that there exists a saddle point for the function $\phi_t(x, y)$ for all nonnegative values of the parameter t. In fact, this saddle point is unique.

Lemma 2.4. For each $t \ge 0$, there exists a unique saddle point (x_t, y_t) of the function $\phi_t(x, y)$ in $\mathscr{X}^0 \times \mathscr{Y}^0$.

Proof. We discussed already the existence. For all $t \ge 0$, $\phi_t(x, y)$ is a strictly convex function of x and a strictly concave function of y. Further, $\mathscr{X}^0$ and $\mathscr{Y}^0$ are nonempty, bounded sets by our assumptions. Therefore, for each $\hat{y} \in \mathscr{Y}^0$, there is a unique minimizer of $\phi_t(x, \hat{y})$ in $\mathscr{X}^0$, and for each $\hat{x} \in \mathscr{X}^0$ there is a unique maximizer of $\phi_t(\hat{x}, y)$ in $\mathscr{Y}^0$. Consequently, if

(x^1, y^1) and (x^2, y^2) are two different saddle points of $\phi_t(x, y)$, we must have that $x^1 \neq x^2$ and $y^1 \neq y^2$. But now the strict convexity/concavity properties of $\phi_t(x, y)$ and the saddle-point properties of (x^1, y^1) and (x^2, y^2) lead to the following contradiction:

$$\phi_t(x^2, y^1) < \phi_t(x^2, y^2) < \phi_t(x^1, y^2) < \phi_t(x^1, y^1) < \phi_t(x^2, y^1). \tag{22}$$

Hence, there cannot be multiple saddle points of ϕ_t. □

As in the previous subsection, we have a set of optimality conditions that characterize the saddle points of the functions ϕ_t for $t \geq 0$. The pair (x_t, y_t) is a saddle point of the function $\phi_t(x, y)$ if and only if there exist Lagrange multipliers $\lambda_t \in \mathbb{R}^m$, $\delta_t^L, \delta_t^U \in \mathbb{R}^n$ and $V_t^L, V_t^U, Z_t \in \mathscr{S}^n$ that satisfy conditions identical to (15)–(17), except that loose inequalities are replaced by strict inequalities and that the complementarity equations appearing at the bottom of (15)–(17) are replaced with the following relations:

$$(Ax_t - b) \circ \lambda_t = \mu_t e, \tag{23}$$

$$(c_t - c^L) \circ d_t^L = \mu_t e, \tag{24a}$$

$$(c^U - c_t) \circ d_t^U = \mu_t e, \tag{24b}$$

$$(Q_t - Q^L) \circ V_t^L = \mu_t E, \tag{25a}$$

$$(Q^U - Q_t) \circ V_t^U = \mu_t E, \tag{25b}$$

$$Q_t Z_t = \mu_t I. \tag{25c}$$

Above, $\mu_t = 1/t$, e denotes a vector of ones of appropriate dimension, and E denotes an $n \times n$ matrix with 1 on the diagonal entries and 1/2 on the offdiagonal entries. E is the symmetrization of a lower or upper triangular matrix of ones.

Equations (23)–(25) can be viewed as a perturbation of the complementarity equations in the original system (15)–(17). Instead of insisting that the complementary quantities have a zero product, we now insist that these products are equal for all complementary pairs.

As Lemma 2.4 indicates, there exists a unique solution to the system (23)–(25) coupled with the feasibility equations for each nonnegative t. The set of solutions to this system for different values of t defines the central path for our saddle-point problem,

$$\mathscr{C} := \{(x_t, c_t, Q_t)\ \exists t \geq 0, \text{ s.t. } (x_t, c_t, Q_t) \text{ a saddle point of } \phi_t(x, c, Q)\}. \tag{26}$$

The central path is the main theoretical tool of path-following algorithms, i.e., algorithms that try to reach a solution by generating iterates around the central path for progressively larger values of t. In the next section, we will discuss measures of proximity to the central path and to the saddle points. This discussion will motivate the algorithm that we develop to solve the problem (SPP).

2.3. Proximity Measures and Results on Central Path. In this section, we discuss a global measure of proximity to a saddle point, the duality gap of the function $\phi(x, y)$. We will show that this measure is nicely bounded on the central path $\mathscr{C}$. Then, we introduce measures of proximity to the central path that allow us to show that the duality gap is also bounded at points close to the central path.

Let us start with the following question: Given points $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, how does one determine whether this pair is or is not close to a saddle point? Recall the functions f and g defined in (4) and (10). From Lemma 2.1, we have that

$$f(x) \geq g(y), \qquad \text{for all } x \in \mathscr{X} \text{ and } y \in \mathscr{Y}.$$

Furthermore, (x, y) is a saddle point for $\phi(x, y)$ if and only if $f(x) = g(y)$. Therefore, the difference between these two functions serves as a global measure of proximity to a saddle point,

$$\begin{aligned} \nu(x, y) &:= f(x) - g(y) \\ &= \max_{(c', Q') \in \mathscr{Y}} \phi(x, c', Q') - \min_{x' \in \mathscr{X}} \phi(x', c, Q). \end{aligned} \tag{27}$$

In Ref. 5, Nemirovski calls $\nu(x, y)$ the weak proximity of ϕ. It can be considered also as the duality gap between trial solutions of the primal problem (9) and the dual problem (12). Since g is extended real-valued, so is ν.

The quantities $f(x)$ and $g(y)$ are defined by optimization problems. We can bound the duality gap using feasible solutions to the duals of these optimization problems.

Lemma 2.5. Given $x \in \mathscr{X}$ and $y = (c, Q) \in \hat{\mathscr{Y}}$, let (x, λ), (δ^L, δ^U), (V^L, V^U, Z) be feasible solutions to problems (11), (7), (8) respectively. Then,

$$\begin{aligned} \nu(x, y) \leq{}& (Ax - b)^T \lambda + (c - c^L)^T \delta^L + (c^U - c)^T \delta^U \\ &+ (Q - Q^L) \cdot V^L + (Q^U - Q) \cdot V^U + Q \cdot Z. \end{aligned} \tag{28}$$

Proof. The main tool of this proof is the weak duality theorem. Since we assumed that (x, λ) is feasible for (11), it follows from the weak duality theorem and the equality constraints in (11) that

$$\begin{aligned} g(c, Q) &\geq b^T\lambda - (1/2)\, x^TQx \\ &= -(Ax - b)^T\lambda - (Qx - A^T\lambda)^T x + (1/2)\, x^TQx \\ &= -(Ax - b)^T\lambda + c^Tx + (1/2)\, x^TQx. \end{aligned} \tag{29}$$

Similarly, we have that

$$\begin{aligned} f^c(x) &\leq -(c^L)^T\delta^L + (c^U)^T\delta^U \\ &= (c - c^L)^T\delta^L + (c^U - c)^T\delta^U + c^T(-\delta^L + \delta^U) \\ &= (c - c^L)^T\delta^L + (c^U - c)^T\delta^U + c^Tx \end{aligned} \tag{30}$$

and that

$$\begin{aligned} f^Q(x) &\leq -(Q^L)\cdot V^L + (Q^U)\cdot V^U \\ &= (Q - Q^L)\cdot V^L + (Q^U - Q)\cdot V^U + Q\cdot(-V^L + V^U) \\ &= (Q - Q^L)\cdot V^L + (Q^U - Q)\cdot V^U + Q\cdot Z + Q\cdot(1/2)xx^T. \end{aligned} \tag{31}$$

Combining (29), (30), (31), we obtain

$$\begin{aligned} v(x, y) &= f(x) - g(y) \\ &= f^c(x) + f^Q(x) - g(c, Q) \\ &\leq (Ax - b)^T\lambda + (c - c^L)^T\delta^L + (c^U - c)^T\delta^U \\ &\quad + (Q - Q^L)\cdot V^L + (Q^U - Q)\cdot V^U + Q\cdot Z, \end{aligned}$$

as required. □

Note that the bound in Lemma 2.5 depends on the particular choice of the feasible variables of the corresponding dual problems. A better choice of these variables will lead to better bounds. In particular, one can choose values that optimize or nearly optimize the corresponding problems to get the tightest possible bounds.

On the central path, the right-hand side of the inequality (28) takes a much simpler form. From now on, we use μ_t (the standard notation for the barrier problem parameter) to denote $1/t$, the reciprocal of our barrier problem parameter.

Lemma 2.6. For a point $(x_t, y_t) \in \mathscr{C}$, the following inequality holds:

$$v(x_t, y_t) \leq (n^2 + 4n + m)\mu_t. \tag{32}$$

Proof. Note that the sets of variables (x_t, λ_t), (δ_t^L, δ_t^U), (V_t^L, V_t^U, Z_t) that satisfy the systems of Eqs. (23)–(25) are feasible solutions to problems (11), (7), (8). Therefore, (28) holds for $(x, y) = (x_t, y_t)$. Since the expression on the right-hand side of (28) can be obtained by simply adding Eqs. (23)–(25), using Lemma 2.5 and recalling the description of the matrix E following Eq. (25), we obtain the desired result. □

Lemma 2.6 is our motivation for developing an algorithm that follows the central path to solve the problem (SPP). For points on the central path, the duality gap converges to zero as the parameter t is increased. However, it is often very hard to find points that are exactly on the central path and we would like to develop a version of Lemma 2.6 for points that are close to the central path in some well-defined sense. First, for notational convenience, we introduce a set of slack/surplus variables as follows:

$$w = A\,x - b, \tag{33a}$$

$$r^L = c - c^L, \qquad r^U = c^u - c, \tag{33b}$$

$$S^L = Q - Q^L, \qquad S^U = Q^U - Q. \tag{33c}$$

The following primal-dual measure of proximity to the central path is a generalization of the measures used in interior-point methods for linear and semidefinite programming and is also similar to the one suggested by Sun, Zhu, and Zhao in Ref. 12:

$$\Delta(P, D, t) = t\left[\begin{array}{l} \sum_{i=1}^{m} (w_i\lambda_i - \mu_t)^2 \\ + \sum_{j=1}^{n} (r_i^L\delta_i^L - \mu_t)^2 + \sum_{j=1}^{n} (r_i^U\delta_i^U - \mu_t)^2 \\ + \sum_{i=1}^{n} (S_{jj}^L V_{jj}^L - \mu_t)^2 + \sum_{1\le i<j\le n} (2S_{ij}^L V_{ij}^L - \mu_t)^2 \\ + \sum_{j=1}^{n} (S_{jj}^U V_{jj}^U - \mu_t)^2 + \sum_{1\le i<j\le n} (2S_{ij}^U V_{ij}^U - \mu_t)^2 \\ + \|QZ - \mu_t I\|_F^2 \end{array}\right]^{1/2}$$

$$= t\left[\left\|\begin{array}{l} w \circ \lambda - \mu_t e_m \\ r^L \circ \delta^L - \mu_t e_n \\ r^U \circ \delta^U - \mu_t e_n \\ \mathrm{svec}(S^L) \circ \mathrm{svec}(V^L) - \mu_t e_{n(n+1)/2} \\ \mathrm{svec}(S^U) \circ \mathrm{svec}(V^U) - \mu_t e_{n(n+1)/2} \end{array}\right\|^2 + \|QZ - \mu_t I\|_F^2\right]^{1/2}. \tag{34}$$

Above,

$$P = (x, Q, \lambda, w, r^l, r^u, S^L, S^U)$$

denotes the primal variables and slacks and

$$D = (\delta^L, \delta^U, V^L, V^U, Z)$$

denotes the corresponding dual variables. The operator svec in the equation above vectorizes a given $n \times n$ symmetric matrix U as follows:

$$\operatorname{svec}(U) := (u_{11}, \sqrt{2}u_{21}, \ldots, \sqrt{2}u_{n1}, u_{22}, \sqrt{2}u_{32}, \ldots, \sqrt{2}u_{n2}, \ldots, u_{nn})^T. \quad (35)$$

The factor $\sqrt{2}$ is introduced so that svec is an isometry between $\mathscr{S}^{n\times n}$ and $\mathbb{R}^{n(n+1)/2}$ with their respective standard inner products. This operator was discussed, e.g., in Ref. 8.

It is easy to verify that the measure Δ is zero for only points on the central path with $x = x_t$, etc. When this local measure is small, we can bound the global proximity measure ν as indicated in the lemma below.

Lemma 2.7. Given $x \in \mathscr{X}$ and $y = (c, Q) \in \mathscr{Y}$, let (x, λ), (δ^L, δ^U), (V^L, V^U, Z) be feasible solutions to problems (11), (7), (8) respectively. Further, let w, r^L, r^U, S^L, S^U be as in (33) and let

$$P = (x, Q, \lambda, w, r^l, r^u, S^L, S^U) \quad \text{and} \quad D = (\delta^L, \delta^U, V^L, V^U, Z).$$

Then,

$$\Delta(P, D, t) \leq \theta \quad (36)$$

implies that

$$\nu(x, y) \leq (n^2 + 4n + m)\,(1 + \theta/\sqrt{n^2 + 4n + m})\mu_t. \quad (37)$$

Proof. From Lemma 2.5, we have that

$$\nu(x, y) \leq w^T\lambda + (r^L)^T\delta^L + (r^U)^T\delta^U + S^L \cdot V^L + S^U \cdot V^U + Q \cdot Z.$$

For a given $n \times n$ matrix A, let diag(A) denote the n-dimensional vector consisting of the diagonal entries of the matrix A. Then,

$$Q \cdot Z = \operatorname{trace}(QZ) = e_n^T \operatorname{diag}(QZ).$$

Now, observe that

$$w^T\lambda + (r^L)^T\delta^L + (r^U)^T\delta^U + S^L \cdot V^L + S^U \cdot V^U + Q \cdot Z$$

$$= e_r^T \begin{bmatrix} w \circ \lambda \\ r^L \circ \delta^L \\ r^U \circ \delta^U \\ \mathrm{svec}(S^L) \circ \mathrm{svec}(V^L) \\ \mathrm{svec}(S^U) \circ \mathrm{svec}(V^U) \\ \mathrm{diag}(QZ) \end{bmatrix},$$

with

$$r = n^2 + 4n + m.$$

Let us denote the long vector on the right-hand side of the above equation with ξ. Now, using the Cauchy–Schwarz inequality and the triangle inequality, we have

$$\begin{aligned} \nu(x, y) &\leq \|e_r\|\,\|\xi\| \leq \sqrt{r}(\|\xi - \mu_t e_r\| + \mu_t\|e_r\|) \\ &\leq \sqrt{n^2 + 4n + m}(\theta + \sqrt{n^2 + 4n + m})\mu_t \\ &= (n^2 + 4n + m)\,(1 + \theta/\sqrt{n^2 + 4n + m})\mu_t. \end{aligned}$$

Since

$$\|\mathrm{diag}(QZ) - \mu_t e\| \leq \|QZ - \mu_t I\|_F,$$

we have that

$$t\|\xi - \mu_t e_r\| \leq \Delta(P, D, t) \leq \theta;$$

therefore, the last inequality above is also justified. □

Measures like Δ have been used for describing the 2-norm neighborhoods of the central path in convex optimization problems and also in the development of short-step path-following algorithms. It is possible to develop predictor-corrector type algorithms using the measure Δ by generating iterates that satisfy inequality (36) for two alternating values of θ: a smaller value for the corrector steps and a larger value for the predictor steps. However, the level of nonlinearity in our problem makes the extension of standard complexity analyses for such algorithms a nontrivial task. We will not pursue such algorithms any further in this study and instead focus on a specialization of the Nemirovski approach for saddle-point problems outlined in Ref. 5.

In Ref. 5, a fundamental work on self-concordant convex–concave functions, Nemirovski extends some of the developments in Ref. 3 to the case of saddle-point problems with convex–concave functions. One of the key elements in this study is a proximity measure that works with only the primal variables.

Recall the definition of the barrier functions for the sets $\mathscr{X}$ and $\mathscr{Y}$ given in (18) and (19). Also, recall that $F(x)$ is a self-concordant barrier for $\mathscr{X}$ with parameter m and that $G(y)$ is a self-concordant barrier for $\mathscr{Y}$ with parameter n^2+4n. Finally, recall the saddle-barrier function ϕ_t defined in (20),

$$\begin{aligned}\phi_t(x, y) &= \phi_t(x, c, Q)\\ &= t\phi(x, y) + F(x) - G(y),\end{aligned} \tag{38}$$

for $t \geq 0$. Given $(\hat{x}, \hat{y}) \in \mathscr{X} \times \mathscr{Y}$, we define the following restricted functions:

$$\phi_{\hat{y}}^t(x) := t\phi(x, \hat{y}) + F(x), \qquad \forall x \in \mathscr{X}^0, \tag{39}$$

$$\phi_{\hat{x}}^t(y) := t\phi(\hat{x}, y) - G(y), \qquad \forall y \in \mathscr{Y}^0. \tag{40}$$

Note that $\phi_{\hat{y}}^t(x)$ is a strictly convex function of $x \in \mathscr{X}^0$ and $\phi_{\hat{x}}^t(y)$ is a strictly concave function of $y \in \mathscr{Y}^0$, as long as $t \geq 0$. Furthermore, since $\phi(x, \hat{y})$ is a convex quadratic function of x when $\hat{y} \in \mathscr{Y}$, and since $\phi(\hat{x}, y)$ is a linear function of y, using Proposition 3.2.1 (ii) and Proposition 3.2.2 of Ref. 3, we conclude that $\phi_{\hat{y}}^t(x)$ and $-\phi_{\hat{x}}^t(y)$ are strongly self-concordant functions in their domains.

The magnitude of the progress made by a Newton step for minimizing a given convex function $\psi(z)$ can serve as a measure of proximity to the minimizer of the function. The motivation behind such a measure is the expectation that Newton steps will generate bigger improvements away from a solution and only marginal improvements close to a solution, where improvements are measured in an absolute sense.

Given a strictly convex function $\psi(\cdot)$ and a vector z in its domain for which the Hessian $\nabla^2\psi(z)$ is nonsingular, consider the following function:

$$\eta(\psi, z) := \sqrt{\nabla\psi^T(z)[\nabla^2\psi(z)]^{-1}\nabla\psi(z)}. \tag{41}$$

The quantity in the square-root on the right-hand side of the above equation is the quadratic Taylor series approximation to the decrease in the value of the function ψ by taking a full Newton step from the point z. In Ref. 3, Nesterov and Nemirovski call $\eta(\psi, z)$ the Newton decrement. In Ref. 5, Nemirovski considers a generalization of the Newton decrement for convex–concave functions. This generalized measure can be represented as follows in our case:

$$\eta(\phi_t, x, y) := \sqrt{\eta^2(\phi_y^t, x) + \eta^2(-\phi_x^t, y)}, \tag{42}$$

where

$$t > 0 \quad \text{and} \quad (x, y) \in \mathscr{X}^0 \times \mathscr{Y}^0.$$

Observe that the Hessian matrices $\nabla^2 \phi_y^t(x)$ and $-\nabla^2 \phi_x^t(y)$ are positive definite for all $(x,y) \in \mathscr{X}^0 \times \mathscr{Y}^0$. We are using the function η with two arguments to denote the Newton decrement for a self-concordant function and the function η with three arguments for the generalized Newton decrement for convex–concave saddle functions; no confusion should arise. Note that neither $\eta(\psi, z)$ nor $\eta(\phi, x, y)$ is scale invariant; in fact,

$$\eta(\alpha\psi, z) = \sqrt{\alpha}\,\eta(\psi, z),$$

and similarly for $\eta(\phi_t, x, y)$. We will use $\eta(\phi_t, x, y)$ as a measure of proximity to the central path $\mathscr{C}$. First, we observe that this measure vanishes only on the central path.

Proposition 2.1. For $t \geq 0$ and $(\hat{x}, \hat{y}) \in \mathscr{X}^0 \times \mathscr{Y}^0$, the Newton decrement $\eta(\phi_t, \hat{x}, \hat{y}) = 0$ if and only if $\hat{x} = x_t$, $\hat{y} = y_t$, where (x_t, y_t) are as in the solution of (23)–(25).

Proof. First, assume that

$$\eta(\phi_t, \hat{x}, \hat{y}) = 0, \qquad \text{for some } t \geq 0 \text{ and } (\hat{x}, \hat{y}) \in \mathscr{X}^0 \times \mathscr{Y}^0.$$

Since $\phi_{\hat{y}}^t(x)$ is a strictly convex function of $x \in \mathscr{X}^0$, and since $-\phi_{\hat{x}}^t(y)$ is a strictly convex function of $y \in \mathscr{Y}^0$ for $t \geq 0$, each with positive-definite Hessian,

$$\eta(\phi_t, \hat{x}, \hat{y}) = 0$$

implies that

$$\nabla \phi_{\hat{y}}^t(\hat{x}) = 0 \qquad \text{and} \qquad \nabla \phi_{\hat{x}}^t(\hat{y}) = 0.$$

In turn, this indicates that $\hat{x}$ minimizes $\phi_{\hat{y}}^t(\cdot)$ and that $\hat{y}$ maximizes $\phi_{\hat{x}}^t(\cdot)$. So, $(\hat{x}, \hat{y})$ is a saddle point for $\phi_t(x, y)$. Since Lemma 2.4 indicates that ϕ_t has a unique saddle point for each $t \geq 0$, $(\hat{x}, \hat{y})$ must be this saddle point and solves (23)–(25).

On the other hand, if (x_t, y_t) solves (23)–(25), it follows easily that

$$\nabla \phi_{y_t}^t(x_t) = 0,$$

$$\nabla \phi_{x_t}^t(y_t) = 0.$$

Therefore, $\eta(\phi_t, x_t, y_t) = 0$. □

We can prove also that, if we are close to the central path with respect to the proximity measure $\eta(\phi^t, x, y)$, then we are close to a saddle point.

Lemma 2.8. If $(\bar{x}, \bar{y}) \in \mathscr{X} \times \mathscr{Y}$ satisfies $\eta(\phi_t, \bar{x}, \bar{y}) \leq \beta$ with $\beta \leq 1/2$, then

$$
\begin{aligned}
v(\bar{x}, \bar{y}) &= f(\bar{x}) - g(\bar{y}) \\
&\leq (1 + 6\beta/\sqrt{n^2 + 4n + m})(n^2 + 4n + m)\mu_t.
\end{aligned}
\tag{43}
$$

Proof. From the assumption that $\eta(\phi_t, \bar{x}, \bar{y}) \leq \beta$ and Eq. (41), it follows that there exists β_x and β_y satisfying

$$
\eta(\phi^t_{\bar{y}}, \bar{x}) \leq \beta_x \qquad \text{and} \qquad \eta(-\phi^t_{\bar{x}}, \bar{y}) \leq \beta_y,
$$

as well as

$$
\beta_x^2 + \beta_y^2 \leq \beta^2.
$$

Let us fix $\bar{y}$ and consider

$$
\phi_{\bar{y}}(x) := \phi(x, \bar{y}). \tag{44}
$$

Also, recall

$$
\phi^t_{\bar{y}}(x) = t\phi(x, \bar{y}) + F(x).
$$

We argued above that $\phi^t_{\bar{y}}(x)$ is a strictly convex function for every $\bar{y} \in \mathscr{Y}$. Because of the barrier property of the function $F(x)$, we have that, for every point x on the boundary of $\mathscr{X}$ and every strictly feasible sequence $\{x_k\}$ such that $\{x_k\} \to x$, we have $\phi^t_{\bar{y}}(x_k) \to \infty$. Further, since $\bar{y} \in \mathscr{Y}^0$, the strictly convex quadratic function $\phi_{\bar{y}}(x)$ is bounded below and grows faster than the logarithmic barrier function $F(x)$ along divergent feasible directions. Therefore, the minimum of $\phi^t_{\bar{y}}(x)$ is achieved. Indeed, there exists a unique minimizer, say $\hat{x}^t \in \mathscr{X}^0$, of the function $\phi^t_{\bar{y}}(x)$. Then, we have that

$$
\nabla \phi^t_{\bar{y}}(\hat{x}^t) = 0 \quad \text{or} \quad t\nabla\phi_{\bar{y}}(\hat{x}^t) + \nabla F(\hat{x}^t) = 0. \tag{45}
$$

The convex function $\phi_{\bar{y}}$ is always underestimated by the first-order Taylor approximation. Therefore, for any $x \in \mathscr{X}$, we have

$$
\begin{aligned}
\phi_{\bar{y}}(x) = \phi(x, \bar{y}) &\geq \phi_{\bar{y}}(\hat{x}^t) + (x - \hat{x}^t)^T \nabla \phi_{\bar{y}}(\hat{x}^t) \\
&= \phi_{\bar{y}}(\hat{x}^t) - \mu_t (x - \hat{x}^t)^T \nabla F(\hat{x}^t),
\end{aligned}
$$

where the equality follows from (45). Furthermore, since F is a self-concordant barrier function for $\mathscr{X}$ with parameter m, we have that

$$
(x - \hat{x}^t)^T \nabla F(\hat{x}^t) \leq m;
$$

see Eq. (2.3.2) in Ref. 3, p. 34. Now,

$$\phi(x, \bar{y}) = \phi_{\bar{y}}(x) \geq \phi_{\bar{y}}(\hat{x}^t) - m\mu_t, \qquad \forall x \in \mathscr{X},$$

and therefore,

$$\begin{aligned} g(\bar{y}) &= \min_{x \in \mathscr{X}} \phi(x, \bar{y}) \\ &\geq \phi_{\bar{y}}(\hat{x}^t) - m\mu_t. \end{aligned} \tag{46}$$

Again, since F is a self-concordant barrier function for $\mathscr{X}$ with parameter m, we have that, by definition,

$$|\nabla F(x)^T h| \leq \sqrt{m}\sqrt{h^T \nabla^2 F(x) h}.$$

Therefore,

$$\begin{aligned} F(\hat{x}^t) - F(\bar{x}) &\leq (\bar{x} - \hat{x}^t)^T \nabla F(\hat{x}^t) \\ &\leq \sqrt{m}\sqrt{(\bar{x} - \hat{x}^t)^T \nabla^2 F(\hat{x}^t)(\bar{x} - \hat{x}^t)}. \end{aligned}$$

Furthermore, since the function $\phi_{\bar{y}}^t$ is also self concordant [see comments following Eqs. (39) and (40)], using inequality (2.16) in Ref. 13, or equivalently (79) in Ref. 5, we obtain

$$\begin{aligned} \phi_{\bar{y}}^t(\bar{x}) - \phi_{\bar{y}}^t(\hat{x}^t) &\leq -\eta(\phi_{\bar{y}}^t, \bar{x}) - \log[1 - \eta(\phi_{\bar{y}}^t, \bar{x})] \\ &\leq -\beta_x - \log(1 - \beta_x) \\ &\leq \beta_x. \end{aligned}$$

The first inequality holds since $-x - \log(1 - x)$ is an increasing function for $x \in [0, 1)$, and the second inequality holds since $\beta_x \leq \beta \leq 1/2$. Similarly, using (2.17) in Ref. 13 and $\beta_x \leq 1/2$, we have also

$$\begin{aligned} \sqrt{(\bar{x} - \hat{x}^t)^T \nabla^2 \phi_{\bar{y}}^t(\hat{x}^t)(\bar{x} - \hat{x}^t)} &\leq \eta(\phi_{\bar{y}}^t, \bar{x})/[1 - \eta(\phi_{\bar{y}}^t, \bar{x})] \\ &\leq \beta_x/(1 - \beta_x) \leq 2\beta_x. \end{aligned}$$

Now, combining the above results, we obtain

$$\begin{aligned} \phi_{\bar{y}}(\bar{x}) - \phi_{\bar{y}}(\hat{x}^t) &= \mu_t[\phi_{\bar{y}}^t(\bar{x}) - \phi_{\bar{y}}^t(\hat{x}^t)] + \mu_t[F(\hat{x}^t) - F(\bar{x})] \\ &\leq \beta_x\mu_t + \sqrt{m}\sqrt{(\bar{x} - \hat{x}^t)^T \nabla^2 F(\hat{x}^t)(\bar{x} - \hat{x}^t)}\mu_t \\ &\leq \beta_x\mu_t + \sqrt{m}\sqrt{(\bar{x} - \hat{x}^t)^T \nabla^2 \phi_{\bar{y}}^t(\hat{x}^t)(\bar{x} - \hat{x}^t)}\mu_t \\ &\leq (1 + 2\sqrt{m})\beta_x\mu_t. \end{aligned}$$

The second inequality above holds, since

$$\nabla^2 F(\hat{x}^t) \preccurlyeq \nabla^2 \phi_{\bar{y}}^t(\hat{x}^t).$$

The final inequality combined with (46) indicates that

$$\begin{aligned}\phi(\bar{x}, \bar{y}) - g(\bar{y}) &\leq m\mu_t + (1 + 2\sqrt{m})\beta_x \mu_t \\ &\leq (1 + 3\beta_x/\sqrt{m}) m\mu_t. \end{aligned} \tag{47}$$

After fixing $\bar{x}$, a symmetric argument yields

$$f(\bar{x}) - \phi(\bar{x}, \bar{y}) \leq (1 + 3\beta_y/\sqrt{n^2 + 4n})(n^2 + 4n)\mu_t. \tag{48}$$

Simple algebra with (47) and (48) produces the desired result. □

Now, we have the necessary machinery to present an algorithm. We saw that the points on the central path $\mathscr{C}$ approach the set of saddle points for the function ϕ. Further, we described two different ways of measuring the proximity of trial solutions to the points on the central path. The next section will introduce an algorithm that generates iterates that are close to the central path in terms of the measure η described above and are progressively closer to the set of saddle points for the function ϕ.

3. Algorithm and Its Analysis

3.1. Interior-Point Algorithm. We propose a short-step algorithm below to find a saddle point of problem (SPP). This algorithm can be viewed as a specialization of the short-step algorithm proposed in Ref. 5 and is also related to the short-step path-following method for variational inequalities described in Chapter 7 of Ref. 3. The method in Ref. 3 updates the central path parameter t according to the formula

$$t_+ = (1 + \delta/\sqrt{\theta})t,$$

where θ is the parameter of the barrier function for the domain of the problem ($n^2 + 4n + m$ in our case) and δ is a small constant ($\delta \leq 0.01$ in our case). Then, the method of Ref. 3 uses a single Newton step to find the new iterate satisfying a proximity bound. In Ref. 5, Nemirovski develops an alternative method that can replace δ above with a larger constant such as 1, but requires an inner iteration procedure (the so-called saddle Newton method in Ref. 5, Section 4.2), which may take several steps, but a bounded number of steps, to generate the next iterate.

Step 1. Initialization. Choose α and β that satisfy the relationships

$$\gamma := 1.3[(1 + \alpha)\beta + \alpha\sqrt{n^2 + 4n + m}] < 1,$$

$$\gamma^2(1 + \gamma)/(1 - \gamma) \leq 1.3\beta.$$

Find $t_0 > 0$ and $(x_0, y_0) \in \mathscr{X}^0 \times \mathscr{Y}^0$ satisfying $\eta(\phi_{t_0}, x_0, y_0) \leq \beta$. Set $k = 0$.

Step 2. Iteration k. Check the inequality

$$t_k < (1/\epsilon)(1 + 6\beta/\sqrt{n^2 + 4n + m})(n^2 + 4n + m).$$

If satisfied, stop. Otherwise, go to Step 3.

Step 3. Update. Set

$$t_{k+1} = (1 + \alpha)t_k.$$

Take a full Newton step,

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - [\nabla^2 \phi_{t_{k+1}}(x_k, y_k)]^{-1} \nabla \phi_{t_{k+1}}(x_k, y_k).$$

Set $k = k + 1$. Return to Step 2.

Our method and analysis offers a compromise between these two approaches. We improve the constant δ to at least 0.1 from 0.01, but still use a single Newton step between parameter updates as in the method in Chapter 7 of Ref. 3. We present a significantly simpler algorithm than the one presented in Ref. 5 and show that it suffices in our case to take a single Newton step to generate a new iterate with a small Newton decrement. Because of the inherent structure and properties of our problem, we can give a more immediate reasoning for its complexity bounds.

Note that, for example, α and β can be chosen as

$$\alpha = 0.1/\sqrt{n^2 + 4n + m} \quad \text{and} \quad \beta = 0.1$$

to satisfy the condition in the initialization step.

3.2. Polynomiality of the Algorithm. The complexity analysis of the saddle-point algorithm of Section 3.1 centers on the Newton decrement, namely the measure $\eta(\phi_t, x, y)$ that we introduced in the previous section. We have shown already that, if the Newton decrement is small, then the iterate is close to the central path in Lemma 2.8. The main task in this section is to show that, when the central path parameter t is increased slightly, a full Newton step will be admissible and will generate an iterate that also has a small Newton decrement (Theorem 3.2). The combination of these two results will lead to the polynomial convergence conclusion (Theorem 3.3).

As mentioned above, our algorithm can be viewed as a specialization of the algorithm in Ref. 5. Consequently, our analysis borrows from the analysis in this work as well. For example, as in Proposition 5.3 of Ref. 5, we will show that our saddle-barrier functions, when multiplied with a suitable constant, are self-concordant convex–concave functions; see Theorem 3.1.

Let us start by defining the necessary terms. Nesterov and Nemirovski (Ref. 3) showed that a large class of convex optimization problems can be solved in polynomial time by introducing the notion of a self-concordant function. The defining property of self-concordant functions is that their third derivative can be bounded in a particular way by the second derivative, so that the second derivative does not change rapidly. For such functions, second-order approximations are satisfactory, Newton steps can be taken confidently and lead to significant improvements. If the feasible set of the problem at hand has an associated self-concordant barrier function, polynomially convergent algorithms can be developed readily.

For convex–concave saddle functions, Nemirovski (Ref. 5) introduced a similar notion of self-concordant convex–concave (s.c.c.c.) functions. The following is Definition 2.2 of Ref. 5.

Definition 3.1. Let X, Y be open convex domains in R^n and R^m, and let

$$f(x, y) \colon X \times Y \to R$$

be a C^3 function. We say that the function is s.c.c.c if f is convex in X for every $y \in Y$, concave in $y \in Y$ for every $x \in X$, and subject to (i) and (ii) below:

(i) for every $x \in X$, $-f(x, \cdot)$ is a barrier for Y, and for every $y \in Y, f(\cdot, y)$ is a barrier for X;

(ii) for every $z = (x, y) \in X \times Y$ and for every $dz = (dx, dy) \in R^n \times R^m$,

$$|D^3 f(z)[dz, dz, dz]| \leq 2[dz^T S_f(z) dz]^{3/2}, \tag{49}$$

where

$$S_f(z) = \begin{bmatrix} \nabla^2_{xx} f(z) & 0 \\ 0 & -\nabla^2_{yy} f(z) \end{bmatrix}.$$

We see that, if $F(x)$ is a self-concordant barrier for X and $G(y)$ is a self-concordant barrier for Y, then $F(x) - G(y)$ is a s.c.c.c. barrier for $X \times Y$. Conversely, if $f(x, y) \colon X \times Y \to R$ is a s.c.c.c. function, then the function $f(\cdot, y)$ is self-concordant on X for every $y \in Y$ and the function $-f(x, \cdot)$ is self-concordant on Y for every $x \in X$. Similar to the case of self-concordant functions, we have that the Newton method for saddle functions will have quadratic local convergence if the saddle function is s.c.c.c.; see Ref. 5, Theorem 4.1.

We will show that, for all $t \geq 0$ and for a properly chosen $\Gamma \geq 1$, the saddle-barrier functions $\Gamma\phi_t(x, y)$ are self-concordant convex–concave functions. To obtain this conclusion, we need to bound the third derivative of these functions in terms of their second derivatives. Since we know already that the barrier part $F(x) - G(y)$ of these functions is s.c.c.c., all we need is to show that the function $\phi(x, y)$ is compatible with the self concordance of the barrier terms in a well-defined manner. Lemma 3.1 below establishes this result. In Lemma 3.1 and later in the paper, for a given positive-definite matrix B, we use the notation $\|u\|_B$ to denote the induced norm, i.e.,

$$\|u\|_B = \sqrt{u^T B u}.$$

We also use $G''(\cdot)$ as a short-hand notation for the Hessian of G.

Lemma 3.1. For every $(x, y) \in \mathscr{X}^0 \times \mathscr{Y}^0$ and $h = (u, v) \in \mathbb{R}^n \times (\mathbb{R}^n \times \mathscr{S}^n)$, the function

$$\begin{aligned}\phi(x, y) &= \phi(x, c, Q)\\ &= c^T x + (1/2) x^T Q x\end{aligned}$$

satisfies the following inequality:

$$|D^3\phi(x, y)[h, h, h]| \leq 3u^T Q u \|v\|_{G''(y)}.$$

Proof. We start by evaluating the differentials of the function ϕ. Let $v_c \in \mathbb{R}^n$, $V_Q \in \mathscr{S}^n$ be such that $v = (v_c, V_Q)$. Then, we have

$$\begin{aligned}
D\phi(x, c, Q)[h] &= c^T u + v_c^T x + u^T Q x + (1/2) x^T V_Q x,\\
D^2\phi(x, c, Q)[h, h] &= 2v_c^T u + u^T Q u + 2u^T V_Q x,\\
D^3\phi(x, c, Q)[h, h, h] &= 3u^T V_Q u.
\end{aligned}$$

Next, defining

$$\begin{aligned}
\pi &:= \pi_y(v_c, V_Q)\\
&:= \inf\{u \mid (c, Q) \pm u^{-1}(v_c, V_Q) \in \mathscr{Y}\}\\
&\geq \inf\{u \mid Q \pm u^{-1} V_Q \succcurlyeq 0\},
\end{aligned}$$

we have

$$-\pi Q \preccurlyeq V_Q \preccurlyeq \pi Q. \tag{50}$$

Noting that, for the self-concordant function $G(y)$ on $\mathscr{Y}$, the norm $\|\cdot\|_{G''(y)}$ majorizes the function π_y [see Theorem 2.1.1 (ii) in Ref. 3], we conclude that

$$-\|v\|_{G''(y)}Q \preccurlyeq V_Q \preccurlyeq \|v\|_{G''(y)}Q. \tag{51}$$

Putting all the pieces together, we obtain

$$|D^3\phi(x,y)[h,h,h]| = 3|u^T V_Q u| = 3|\langle uu^T, V_Q\rangle|$$

$$\leq 3\pi|u^T Q u| \tag{52}$$

$$\leq 3u^T Qu\|v\|_{G''(y)}. \tag{53}$$

□

According to the terminology introduced in Refs. 3 and 5, inequality (52) shows that the function $\phi(x, y)$ is 3-regular on $\mathscr{X} \times \mathscr{Y}$, and inequality (53) shows that $\phi(x, y)$ is 3-compatible with the barrier function $F(x) + G(y)$ on the set $\mathscr{X}^0 \times \mathscr{Y}^0$. Now, we are ready to prove our self-concordance result.

Theorem 3.1. Define $f_t(\cdot,\cdot) = \Gamma\phi_t(\cdot,\cdot)$, where $\Gamma \geq 1.69$. Then, each member of the family $\{f_t(\cdot,\cdot)\}_{t\geq 0}$ is a self-concordant convex–concave function.

Proof. For all $t \geq 0$, each $f_t(\cdot,\cdot)$ satisfies clearly the convexity and barrier properties in Definition 3.1, and it suffices to show that each such function satisfies also (49). As before, consider

$$(x, y) \in \mathscr{X}^0 \times \mathscr{Y}^0 \quad \text{and} \quad h = (u, v) = (u, v_c, V_Q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathscr{S}^n.$$

From Lemma 3.1, we have that

$$|D^3 t\phi(x,y)[h,h,h]| \leq 3tu^T Qu\|v\|_{G''(y)}.$$

Let $a_1, a_2 > 0$ be such that

$$a_1 a_2 = 3;$$

their precise values will be determined later. Using the Hölder inequality on the right-hand side of the expression above, we obtain

$$|D^3 t\phi(x,y)[h,h,h]|$$

$$\leq (2/3)(a_1 tu^T Qu)^{3/2} + (1/3)(a_2\|v\|_{G''(y)})^3. \tag{54}$$

To simplify the notation, let

$$\omega = tu^T Qu, \qquad \zeta = \|u\|_{F''(x)}, \qquad \eta = \|v\|_{G''(y)}.$$

Also, let

$$b_1 = a_1^{3/2}/3 \quad \text{and} \quad b_2 = a_2^3/3.$$

The second inequality below is a direct consequence of the self concordance of F and G on their domains and (54):

$$\begin{aligned} &|D^3 f_t(x, y)[h, h, h]| \\ &\leq \Gamma(|D^3 t\phi(x, y)[h, h, h]| + |D^3 F(x)[u, u, u]| + |D^3 G(y)[v, v, v]|) \\ &\leq \Gamma(2b_1 \omega^{3/2} + b_2 \eta^3 + 2\zeta^3 + 2\eta^3). \end{aligned}$$

Noting that

$$h^T S_{f_t}(x, y) h = \Gamma(\omega + \zeta^2 + \eta^2),$$

the following inequality will imply (49):

$$\Gamma^2[2b_1 \omega^{3/2} + 2\zeta^3 + (2 + b_2)\eta^3]^2 \leq 4\Gamma^3(\omega + \zeta^2 + \eta^2)^3. \tag{55}$$

After expanding and comparing both sides of this expression and using inequalities such as

$$\omega\eta^4 + \omega^2\eta^2 \geq 2\omega^{3/2}\eta^3,$$

one can verify that

$$\begin{aligned} &\Gamma \geq b_1^2, && \Gamma \geq 1, && \Gamma \geq (2 + b_2)^{2/4}, \\ &\Gamma \geq (2 + b_2)/6, && \Gamma \geq b_1/3, && \Gamma \geq b_1(2 + b_2)/6 \end{aligned}$$

form a set of sufficient conditions for (55) to hold. Since a_1 and a_2 (and therefore, b_1 and b_2) were arbitrary except for the condition that $a_1 a_2 = 3$, we can choose them optimally to obtain the weakest bounds on Γ. This is achieved by choosing

$$a_1 \simeq 2.474,$$

which then gives

$$a_2 \simeq 1.213, \qquad b_1 \simeq 1.297, \qquad b_2 \simeq 0.594.$$

For this choice, we see that all values $\Gamma \geq 1.69$ satisfy the sufficient conditions listed above. Therefore, all $\Gamma\phi_t(\cdot, \cdot)$ are self-concordant convex–concave functions as long as $\Gamma \geq 1.69$. □

Theorem 3.2. Let t_i and $(x_i, y_i) \in \mathscr{X}^0 \times \mathscr{Y}^0$ satisfy $\eta(\phi_{t_i}, x_i, y_i) \leq \beta$, where β is chosen as prescribed in the algorithm of Section 3.1. Further, let t_{i+1} and (x_{i+1}, y_{i+1}) be computed as in the same algorithm. Then,

$$\eta(\phi_{t_{i+1}}, x_{i+1}, y_{i+1}) \leq \beta \tag{56}$$

is also satisfied.

Proof. First, we will find a bound on $\eta(\phi_{t_{i+1}}, x_i, y_i)$, i.e., the Newton decrement with the old point and the new barrier parameter. We have

$$\begin{aligned}
\eta(\phi_{y_i}^{t_{i+1}}, x_i) &= \sqrt{\nabla^T \phi_{y_i}^{t_{i+1}}(x_i)[\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i)]^{-1} \nabla \phi_{y_i}^{t_{i+1}}(x_i)} \\
&= \|\nabla \phi_{y_i}^{t_{i+1}}(x_i)\|_{[\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i)]^{-1}} \\
&= \|(1+\alpha)\nabla \phi_{y_i}^{t_i}(x_i) - \alpha \nabla F(x_i)\|_{[\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i)]^{-1}} \\
&\le (1+\alpha)\|\nabla \phi_{y_i}^{t_i}(x_i)\|_{[\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i)]^{-1}} + \alpha \|\nabla F(x_i)\|_{[\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i)]^{-1}} \\
&\le (1+\alpha)\|\nabla \phi_{y_i}^{t_i}(x_i)\|_{[\nabla^2 \phi_{y_i}^{t_i}(x_i)]^{-1}} + \alpha \|\nabla F(x_i)\|_{[\nabla^2 F(x_i)]^{-1}} \\
&= (1+\alpha)\eta(\phi_{y_i}^{t_i}, x_i) + \alpha\sqrt{m}.
\end{aligned}$$

Above, the first inequality follows from the triangle inequality and the second inequality uses the fact that

$$\nabla^2 \phi_{y_i}^{t_{i+1}}(x_i) \succcurlyeq \nabla^2 \phi_{y_i}^{t_i}(x_i) \succcurlyeq \nabla^2 F(x_i),$$

which implies the reversed order for the inverses. The final equality holds, since F is a logarithmically homogeneous self-concordant barrier function with parameter m; see e.g. Proposition 2.3.4 in Ref. 3, p. 41.

A symmetric argument yields

$$\eta(-\phi_{x_i}^{t_{i+1}}, y_i) \le (1+\alpha)\eta(-\phi_{x_i}^{t_i}, y_i) + \alpha\sqrt{n^2+4n}.$$

Combining, we have

$$\begin{aligned}
\eta^2(\phi_{t_{i+1}}, x_i, y_i) \le{}& (1+\alpha)^2[\eta^2(\phi_{y_i}^{t_{i+1}}, x_i) + \eta^2(-\phi_{x_i}^{t_{i+1}}, y_i)] + \alpha^2(n^2+4n+m) \\
&+ 2\alpha(1+\alpha)[\sqrt{m}\,\eta(\phi_{y_i}^{t_{i+1}}, x_i) + \sqrt{n^2+4n}\,\eta(-\phi_{x_i}^{t_{i+1}}, y_i)] \\
\le{}& [(1+\alpha)\eta(\phi_{t_i}, x_i, y_i) + \alpha\sqrt{n^2+4n+m}]^2.
\end{aligned}$$

The second inequality above can be verified by expanding the last term, canceling common terms on both sides of the inequality, squaring the remaining terms, and using the following inequality:

$$(\delta_1+\delta_2)(\gamma_1^2+\gamma_2^2) - (\sqrt{\delta_1}\gamma_1 + \sqrt{\delta_2}\gamma_2)^2 = (\sqrt{\delta_2}\gamma_1 - \sqrt{\delta_1}\gamma_2)^2 \ge 0,$$

with

$$\delta_1 = m, \qquad \delta_2 = n^2+4n, \qquad \gamma_1 = \eta(\phi_{y_i}^{t_{i+1}}, x_i), \qquad \gamma_2 = \eta(-\phi_{x_i}^{t_{i+1}}, y_i).$$

Recall from Section 3.1 that

$$\gamma = (13/10)[(1+\alpha)\beta + \alpha\sqrt{n^2+4n+m}].$$

Since $\eta(\phi_{t_i}, x_i, y_i) \le \beta$, we get

$$\eta(\phi_{t_{i+1}}, x_i, y_i) \le (10/13)\gamma. \tag{57}$$

Consequently, the self-concordant convex–concave function $1.69\phi_{t_{i+1}}(x_i, y_i)$ (see Theorem 3.1) satisfies

$$\eta(1.69\phi_{t_{i+1}}, x_i, y_i) \leq \gamma < 1.$$

Now, we can apply the results in Ref. 5. In particular, using Proposition 2.3.2(a) and Proposition 4.1.4(a) in Ref. 5, we have that our algorithm generates strictly feasible iterates; i.e.,

$$(x_{i+1}, y_{i+1}) \in \mathscr{X}^0 \times \mathscr{Y}^0.$$

Furthermore, from (48) in Ref. 5 as well as our choices for α and β, we get

$$\begin{aligned}\eta(1.69\phi_{t_{i+1}}, x_{i+1}, y_{i+1}) &\leq \gamma^2(1+\gamma)/(1-\gamma)\\ &\leq (13/10)\beta,\end{aligned}$$

and therefore,

$$\begin{aligned}\eta(\phi_{t_{i+1}}, x_{i+1}, y_{i+1}) &\leq (10/13)\gamma^2(1+\gamma)/(1-\gamma)\\ &\leq \beta,\end{aligned}$$

concluding our proof. □

Finally, we present our polynomial complexity result.

Theorem 3.3. The saddle-point algorithm in Section 3.1 finds a feasible point (x, y) with $\nu(x, y) \leq \epsilon$ in $\mathscr{O}(\sqrt{n^2 + 4n + m}\log(1/\epsilon))$ iterations.

Proof. We omit the standard proof which follows from Lemma 2.8 and Theorem 3.2. □

One issue that we did not discuss is the initialization, i.e., how to find $t_0 > 0$ and $(x_0, y_0) \in \mathscr{X}^0 \times \mathscr{Y}^0$ satisfying $\eta(\phi_{t_0}, x_0, y_0) \leq \beta$ to start the algorithm. The algorithm can be initialized by solving approximately the analytic center problems over $\mathscr{X}$ and $\mathscr{Y}$, which can be done in. $\mathscr{O}(\sqrt{n^2 + 4n + m}\log(1/\epsilon))$ time. This will give an approximation, say (x_0, y_0), to the saddle point of the pure barrier function $\phi_0(x, y)$. Then, t_0 can be chosen as the largest t satisfying $\eta(\phi^{t_0}, x_0, y_0) \leq \beta$.

4. Application: Robust Portfolio Optimization

Now, we describe an application of the approach outlined in the previous sections to a problem, that originally motivated this study. In 1952, Markowitz developed a model of portfolio selection that quantified the tradeoff between return and risk, using the expected returns and variances of portfolios (Ref. 14). This model uses estimates of the expected returns on a number of securities with random returns as well as a covariance matrix that describes their interdependencies.

Mathematically, the Markowitz mean-variance optimization problem can be stated as

$$\max_{x \in \mathscr{X}} \mu^T x - \lambda x^T Q x, \tag{58}$$

where μ_i is an estimate of the expected return of security i, q_{ii} is the variance of this return, q_{ij} is the covariance between the returns of securities i and j, λ is a risk-aversion constant. $\mathscr{X}$ is the set of feasible portfolios which may carry information on short-sale restrictions, sector distribution requirements, etc. Since such restrictions are predetermined, we can assume that the set $\mathscr{X}$ is known without any uncertainty at the time the problem is solved. Solving the problem above for different values of λ, one obtains what is known as the efficient frontier of the set of feasible portfolios. The optimal portfolio will be different for individuals with different risk-taking tendencies, but it will be always on the efficient frontier.

One of the limitations of this model is its need to estimate accurately the expected returns and covariances. In Ref. 15, Bawa, Brown, and Klein argue that using estimates of the unknown expected returns and covariances leads to an estimation risk in portfolio choice, and that methods for the optimal selection of portfolio must take into account this risk. Furthermore, the optimal solution is sensitive to perturbations in these input parameters (a small change in the estimate of the return or the variance may lead to a large change in the corresponding solution); see, for example, Refs. 16 and 17. This attribute is unfavorable, since the modeler may want to rebalance periodically the portfolio based on new data and may incur significant transaction costs to do so. Furthermore, using point estimates of the expected return and covariance parameters does not respond to the needs of a conservative investor who does not necessarily trust these estimates and would be more comfortable choosing a portfolio that will perform well under a number of different scenarios. Of course, such an investor cannot expect to get better performance on some of the more likely scenarios, but will have insurance for more extreme cases. All these arguments point to the need of a portfolio optimization formulation that incorporates robustness and tries to find a solution that is relatively insensitive to the inaccuracies in the input data.

For robust portfolio optimization, we propose a model that allows the return and covariance matrix information to be given in the form of intervals. For example, this information may take the form "the expected return on security j is between 8% and 10%" rather than claiming that it is 9%. Mathematically, we will represent this information as membership in the following set:

$$\mathscr{U} = \{(\mu, Q): \mu^L \leq \mu \leq \mu^U, Q^L \leq Q \leq Q^U, Q \succcurlyeq 0\}, \tag{59}$$

where μ^L, μ^U, Q^L, Q^U are the extreme values of the intervals that we just mentioned. The restriction $Q \succcurlyeq 0$ is necessary, since Q is a covariance matrix and therefore must be positive semidefinite. These intervals may be generated in different ways. An extremely cautious modeler may want to use the historical lows and highs of certain input parameters as the range of their values. One may generate different estimates using different scenarios on the general economy and then combine the resulting estimates. Different analysts may produce different estimates for these parameters, and one may choose the extreme estimates as the endpoints of the intervals. One may choose a confidence level and then generate estimates of the covariance and return parameters in the form of prediction intervals.

Given these considerations, the robust optimization problem that we propose is to find a portfolio that maximizes the objective function in (58) in the worst-case realization of the input parameters μ and Q from their uncertainty set $\mathscr{U}$ in (59). Mathematically, this can be written as

$$\max_{x \in \mathscr{X}} \left\{ \min_{(\mu, Q) \in \mathscr{U}} \mu^T x - \lambda x^T Q x \right\}, \tag{60}$$

which is equivalent to

$$\min_{x \in \mathscr{X}} \left\{ \max_{(\mu, Q) \in \mathscr{U}} -\mu^T x + \lambda x^T Q x \right\}.$$

For fixed λ, this problem is exactly in the form (9) and therefore can be solved using the algorithm that we developed in the previous section.

We conclude this section by noting that robust portfolio optimization approaches can be implemented also in the framework of factor models, i.e., when the interdependencies of stock returns are explained through a small number of factors. In Ref. 18, Goldfarb and Iyengar investigate such problems and show that, in this case, the robust portfolio selection problem reduces to a second-order cone programming problem when the uncertainty sets are ellipsoids. Second-order cone problems can be solved efficiently using interior-point approaches similar to the one presented in the previous section.

5. Comments and Conclusions

This study provides a modeling framework where one tries to solve a quadratic programming problem with an uncertain objective function that is known to be a convex function. As the previous section indicates, such modeling environments arise in portfolio optimization. Other situations where this model is appropriate include problems that have an uncertain cost structure that reflects diseconomies-of-scale and hence convexity. Similarly, problems with diminishing returns lead to formulations that are necessarily convex optimization problems even when their inputs are uncertain.

For such environments, we developed a robust optimization strategy. Although it may appear conservative, this strategy is appropriate in situations where the modeler would like to hedge against all possible realizations of the uncertain input parameters. In this sense, robust optimization presents an alternative approach to stochastic programming. In the stochastic optimization framework, one needs to estimate probability distributions for the model which can be a difficult task because of limited data availability. As a result, a stochastic optimization modeler may need to make unjustified assumptions on the distributions and end up overinterpreting the data. Robust optimization approach avoids such pitfalls.

Our formulation and the analysis of the algorithm provides a generalization of some of the previous works on use of interior-point methods for saddle-point problems. We rely on the techniques developed in Refs. 3 and 5 that exploit the properties of self-concordant barrier functions. Future work on the approach introduced here will include the development of an efficient strategy to generate the search directions prescribed by the interior-point algorithm.

References

1. Ben-tal, A., and Nemirovski, A., *Robust Solutions of Uncertain Linear Programs*, Operations Research Letters, Vol. 25, pp. 1–13, 1999.
2. Ben-tal, A., and Nemirovski, A., *Robust Convex Programming*, Mathematics of Operations Research, Vol. 23, pp. 769–805, 1998.
3. Nesterov, Yu., and Nemirovski, A., *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, Pennsylvania, 1994.
4. Lobo, M. S., and Boyd, S., *The Worst-Case Risk of a Portfolio*, Information Systems Laboratory, Stanford University, 1999.
5. Nemirovski, A., *On Self-Concordant Convex–Concave Functions*, Optimization Methods and Software, Vols. 11–12, pp. 303–384, 1999.

6. Nesterov, Yu., and Todd, M. J., *Self-Scaled Barriers and Interior-Point Methods for Convex Programming*, Mathematics of Operations Research, Vol. 22, pp. 1–42, 1997.
7. Nesterov, Yu., and Todd, M. J., *Primal-Dual Interior-Point Methods for Self-Scaled Cones*, SIAM Journal on Optimization, Vol. 8, pp. 324–364, 1998.
8. Todd, M. J., Toh, K. C., and Tütüncü, R. H., *On the Nesterov-Todd Direction in Semidefinite Programming*. SIAM Journal on Optimization, Vol. 8, pp. 769–796, 1998.
9. Hiriart-urruty, J. B., and Lemarechal, C., *Convex Analysis and Minimization Algorithms*, Springer Verlag, Berlin, Germany, 1993.
10. Goffin, J. L., and Vial, J. P., *Convex Nondifferentiable Optimization: A Survey Focused on the Analytic Center Cutting Plane Method*, Technical Report 99.02, HEC/Logilab, University of Geneva, Geneva, Switzerland, 1999.
11. Rockafellar, R. T., *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
12. Sun, J., Zhu, J., and Zhao, G., *A Predictor-Corrector Algorithm for a Class of Nonlinear Saddle-Point Problems*, SIAM Journal on Control and Optimization, Vol. 35, pp. 532–551, 1997.
13. Nemirovski, A., *Interior-Point Polynomial Methods in Convex Programming*, Lecture Notes, Spring Semester 1996, Faculty of Industrial Engineering and Management, Technion, Haifa, Israel, 1996.
14. Markowitz, H., *Portfolio Selection*, Journal of Finance, Vol. 7, pp. 77–91, 1952.
15. Bawa, V. S., Brown, S. J., and Klein, R. W., *Estimation Risk and Optimal Portfolio Choice*, North-Holland, Amsterdam, Netherlands, 1979.
16. Michaud, R. O., *The Markowitz Optimization Enigma: Is Optimized Optimal?*, Financial Analysts Journal, Vol. 45, pp. 31–42, 1989.
17. Michaud, R. O., *Efficient Asset Management*, Harvard Business School Press, Boston, Massachusetts, 1998.
18. Goldfarb, D. and Iyengar, G., *Robust Portfolio Selection Problems*, Technical Report, IEOR Department, Columbia University, 2001.